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Critical Exponents of Fujita Type for Inhomogeneous Parabolic Equations and Systems

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We consider the large-time behavior of sign-changing solutions of inhomogeneous parabolic equations and systems. For example, for $u_t = \Delta u + |u|^p + w(x)$ in $\mathbf{R}^n \times [0, T)$, we show the following: If $n \geq 3$ and $\int_{\mathbf{R}^n} w(x)dx > 0$ and $1 < p \leq n/(n-2)$, then all solutions blow up in finite time, while if $p > n/(n-2)$ there are both global and nonglobal solutions. We show by example that global solutions exist for all $p > 1$ and w satisfying $\int_{\mathbf{R}^n} w(x)dx < 0$. When $n = 1, 2$ and $\int_{\mathbf{R}^n} w(x)dx > 0$, no solution can exist for all time. Extensions of the above result to various geometries and some other problems are indicated. © 2000 Academic Press

1. INTRODUCTION

It has been known since the early 1960s [F] that the long-time existence of nonnegative solutions of a semilinear parabolic equation depends on the relationship between the decay rate of the linear problem (for which diffusion drives the motion) and the blow-up rate for solutions of a related nonlinear ordinary differential equation as follows: When the decay rate

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is larger than the blow-up rate, both global solutions and solutions which blow up in finite time are possible while when the decay rate is smaller than the blow-up rate, all (nontrivial) solutions blow up in finite time. One can find accounts of the historic development of this issue in the survey papers [Le] and [DL]. Recently in a series of papers [Zh1–Zh3] Zhang extended Fujita's result to inhomogeneous Cauchy problems. In this case, the value of the critical exponent and the location of blow-up points are not the same as those for the homogeneous equation. The critical exponent is more closely tied to the critical exponent of the corresponding elliptic problem. For example, the critical exponent $n/(n-2)$ is the same as the infimum of those p for which $\Delta v + v^p = 0$ has singular solutions of the form $v = Ar^{-2/(p-1)}$.

In this paper we prove blow-up and existence results for inhomogeneous equations which are analogous to similar results for homogeneous cases. We show that some of the results can be extended to solutions that may change sign. To our knowledge, blow-up and global existence theorems for such solutions have been obtained in one space dimension, [MY], using $|u|^{p-1}u$ as the extension of u^p to negative values. The current results use the extension to negative values in the form $|u|^p$. There is an extensive literature on blow-up results for wave equations with this nonlinearity as a source term.

We consider solutions of the following three problems in this paper in some detail. Throughout $p > 1$.

$$\begin{aligned} &\Delta u - \partial_t u + |u|^p + w = 0 && \text{in} \quad \mathbf{R}^n \times (0, \infty) \\ &u(x, 0) = u_0(x) && \text{in} \quad \mathbf{R}^n \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\Delta u - \partial_t u + |u|^p = 0 && \text{in} \quad D^c \times (0, \infty) \\ &u(x, t) = f(x) && \text{in} \quad \partial D \times (0, \infty) \\ &u(x, 0) = u_0(x) && \text{in} \quad D^c \end{aligned} \tag{1.2}$$

$$\begin{aligned} &\Delta u - \partial_t u + |u|^p = 0 && \text{in} \quad D^c \times (0, \infty) \\ &\frac{\partial u(x, t)}{\partial n^+} = f(x) && \text{in} \quad \partial D \times (0, \infty) \\ &u(x, 0) = u_0(x) && \text{in} \quad D^c \end{aligned} \tag{1.3}$$

Here D is a bounded domain with smooth boundary and n^+ is the outward (relative to D^c) unit normal of ∂D.

When $D^c = \mathbf{R}^n$, Fujita [F] proved the following for Problem (1.2):

(a) When $1 < p < 1 + 2/n$, Problem (1.2) possesses no global positive solution for $u_0 \geq 0$.

(b) When $p > 1 + 2/n$ and u_0 (≥ 0) is smaller than a small Gaussian, then (1.2) has global positive solutions.

We say that $1 + 2/n$ is critical in the sense of Fujita. When D is a bounded nonempty domain and $f \equiv 0$, it was shown in [BL] that the critical exponent for (1.2) is also $p = 1 + 2/n$. It was shown in both cases that when $p = 1 + 2/n$, the conclusion of (a) holds. See [DL, Le] for precise references.

We introduce the following definition. We assume very mild conditions on the solution as in [Qi].

DEFINITION 1.1. (a) A function $u = u(x, t)$ is a solution of Problem (1.2) in $Q_T \equiv D^c \times [0, T)$ if

(i) $u \in C((0, T); L^1_{\text{loc}}(D^c)) \cap C((0, T); L^p_{\text{loc}}(D^c))$.

(ii) For all compactly supported $\psi \in C^2(D^c \times [0, T)) \cap C^1(\overline{D}^c \times [0, T))$ vanishing on $\partial D \times [0, T)$,

$$\int_0^\tau \int_{D^c} \left[u\Delta\psi + u\partial_t\psi + u^p(y, s)\psi\right] dyds - \int_0^\tau \int_{\partial D} f \frac{\partial\psi}{\partial n^+} dS_y ds - \int_D u(x, .)\psi(x, .)|_0^\tau dx = 0,$$

for all $\tau \in [0, T)$. We assume that n^+ is defined almost everywhere.

(b) A function $u = u(x, t)$ is a solution of (1.3) in $Q_T \equiv D^c \times [0, T)$ if:

(i) $u \in C((0, T); L^1_{\text{loc}}(D^c)) \cap C((0, T); L^p_{\text{loc}}(D^c))$.

(ii) For all compactly supported $\psi \in C^2(D^c \times [0, T)) \cap (C(\overline{D}^c \times [0, T))$,

$$\int_0^\tau \int_{D^c} \left[-\nabla u \nabla\psi + u\partial_t\psi + |u|^p(y, s)\psi\right] dyds + \int_0^\tau \int_{\partial D} f\psi dS_y ds - \int_D u(x, .)\psi(x, .)|_0^\tau dx = 0,$$

for all $\tau \in [0, T)$.

We shall assume that the initial condition u_0 is such that a local solution exists. Here is a sample of the main results of the paper.

THEOREM 2.1. *For Problem* (1.1) *we have the following*:

(a) *If* $p < n/(n-2)$ *and* $\int_{\mathbf{R}^n} w(x)dx > 0$, *then Problem* (1.1) *has no global solutions. If* $n = 1, 2$ *the conclusion holds if* $\int_{\mathbf{R}^n} w(x)dx > 0$ *and* $p > 1$.

(b) *If* $p = n/(n-2)$ *and* $\int_{\mathbf{R}^n} w(x)dx > 0$, $w(x) = O(|x|^{-\epsilon-n})$ *as* $|x| \to \infty$ *for some* $\epsilon > 0$, *and either* $u \geq 0$ *or*

$$\int_{B_R^c(0)} \frac{w^-(y)}{|x-y|^{n-2}} dy = \frac{o(1)}{|x|^{n-2}} \tag{2.1}$$

when R *is large, then Problem* (1.1) *has no global solutions*. (*Here and throughout*, $w^{\pm} = \max(\pm w, 0)$.)

(c) *If* $p > n/(n-2)$, *then for any* $\delta > 0$ *there exists* $\epsilon > 0$ *such that Problem* (1.1) *has global solutions provided that* $|w(x)|, |u_0(x)| \leq \epsilon/(1+|x|^{n+\delta})$ *regardless of whether or not* $\int_{\mathbf{R}^n} w(x)dx > 0$. *There also exists* u_0 *for which the solution is not global for any* w *such that* $\int_{\mathbf{R}^n} w(x)dx > 0$.

(d) *For any* n, $p > 1$, *Problem* (1.1) *has global solutions for some* u_0 *and some* w *such that* $\int_{\mathbf{R}^n} w(x)dx < 0$.

Remark 1.1. The condition for w^- in the theorem is satisfied if w^- decays near infinity faster than $1/|x|^n$. As indicated above, this assumption is unnecessary for nonnegative solutions. If u is nonnegative and does not grow too fast when $|x| \to \infty$, then Theorem 2.1(a) can be proved using Kaplan arguments [BL]. It will be clear from the proof that the decay condition $w(x) = O(|x|^{-\epsilon-n})$ can be relaxed to $w(x) = O(|x|^{-\epsilon-2})$.

The solutions we are discussing can change sign. In case (a) of Theorem 2.1 the solution may or may not eventually become positive. Under the extra assumption on w^- in part (b) we prove below that the solution will become positive at any fixed point (and hence on compact sets) in a finite time depending on this point. However, the solution need not become positive on $\mathbb{R}^n$ in finite time!

It thus becomes difficult to construct subsolutions which blow up in finite time. Here it is shown that $u \notin L^p_{\text{loc}}$ for all time, i.e., the L^p norm of u blows up in finite time on selected compact regions. Since the choice of these regions depends on the equation and boundary conditions, they need to be identified carefully. (We show below that the solutions will generally be positive in a region of the form $\{(x,t) | R \leq |x| \leq 2R, t \geq |x|^{4n}\}$ for all sufficiently large R.) This approach is sufficiently flexible to attack a wide range of problems without any a priori assumptions on the solution. However, it seems limited to dimensions $n \geq 3$. (The proof of Theorem 2.1(a) does not rely upon this argument.)

In the next section we discuss the Cauchy problem for inhomogeneous equations in $\mathbf{R}^n$. In Section 3 we consider inhomogeneous initial boundary value problems in exterior domains, and in Section 4 we examine similar questions for systems. In Section 5 we suggest some avenues for further research.

2. CAUCHY PROBLEM FOR INHOMOGENEOUS EQUATIONS

Here we consider Problem (1.1) and establish Theorem 2.1.

Remark 2.1. The assumption (2.1) on w guarantees that the bounded solutions of $\triangle h(x) + w = 0$ in $\mathbf{R}^n$ satisfy $h(x) \geq c/|x|^{n-2}$ for some positive c and $|x| \to \infty$. For instance, it is satisfied when $w^-(y) \leq C/(1+|y|^{n+\delta})$ for a $\delta > 0$ and $C > 0$. To see this,

$$\begin{aligned}
&\int_{B_R^c(0)} \frac{w^-(y)}{|x-y|^{n-2}} dy \\
&\quad = \int_{|y|\geq R,\, |x|\geq 2|y|} \frac{w^-(y)}{|x-y|^{n-2}} dy + \int_{|y|\geq R,\, |x|\leq 2|y|} \frac{w^-(y)}{|x-y|^{n-2}} dy \\
&\quad \leq \frac{C}{|x|^{n-2}} \int_{|y|\geq R} w^-(y) dy + \frac{C}{|x|^{n-2}} \int_{|y|\geq R,\, |x|\leq 2|y|} \frac{1}{|y|^{2+\delta}|x-y|^{n-2}} dy \\
&\quad = \frac{o(1)}{|x|^{n-2}}.
\end{aligned}$$

Note that no sign condition on the initial value u_0 is needed for part (a).

The following two lemmas will be needed to identify the regions where u is positive in the critical case.

LEMMA 2.1. *Let $n \geq 3$. Suppose h is a solution of $\Delta h(x) + w = 0$, $x \in \mathbf{R}^n$, with $h(x) \to 0$ when $|x| \to \infty$. If $\int_{\mathbf{R}^n} w(x)dx > 0$ and $\int_{B_R^c(0)}(w^-(y)/|x-y|^{n-2})dy = o(1)/|x|^{n-2}$ when R and $|x|$ are large, then there exist positive constants c_1 and r_0 such that $h(x) \geq c_1/|x|^{n-2}$ when $|x| \geq r_0$.*

Proof of Lemma 2.1. Since $\int_{\mathbf{R}^n} w(x)dx > 0$, we can find a ball $B_{R_0}(0)$ such that

$$\int_{B_{R_0}(0)} w^+(y)dy > \int_{B_{R_0}(0)} w^-(y)dy.$$

Then

$$\begin{aligned}
h(x) &= c_n \int_{\mathbf{R}^n} \frac{w(y)}{|x-y|^{n-2}} dy \\
&\geq c_n \int_{B_{R_0}(0)} \frac{w^+(y)}{|x-y|^{n-2}} dy - c_n \int_{B_{R_0}(0)} \frac{w^-(y)}{|x-y|^{n-2}} dy \\
&\quad - c_n \int_{B_{R_0}^c(0)} \frac{w^-(y)}{|x-y|^{n-2}} dy.
\end{aligned}$$

Hence for large x,

$$h(x) \geq \frac{c_n}{(|x|+R_0)^{n-2}}\left(\int_{B_{R_0}(0)} w^+(y)dy\right.$$
$$\left. -\frac{(|x|+R_0)^{n-2}}{(|x|-R_0)^{n-2}}\int_{B_{R_0}(0)} w^-(y)dy\right) + \frac{o(1)}{|x|^{n-2}}.$$

From the hypothesis that $\int_{\mathbf{R}^n} w^+(y)dy > \int_{\mathbf{R}^n} w^-(y)dy$, there exist $\alpha > 1$ and $c_0 > 0$ such that

$$\int_{B_{R_0}(0)} w^+(y)dy - \alpha \int_{B_{R_0}} w^-(y)dy \geq c_0$$

for all R_0 sufficiently large. If $|x|$ is sufficiently large, then $[(|x|+R_0)/(|x|-R_0)]^{n-2} < \alpha$. From the above,

$$h(x) \geq \frac{c_0}{(|x|+R_0)^{n-2}} + \frac{o(1)}{|x|^{n-2}}$$

when $|x| \geq c_1$ for some $c_1 > 0$. Note that c_0 is fixed for all large R_0. Taking $|x|$ and R_0 sufficiently large, $h(x) \geq c/|x|^{n-2}$. ∎

LEMMA 2.2. *Suppose $n \geq 3$. Let $g = g(x,t)$ be the solution of the linear problem*

$$\begin{aligned}&\Delta g - g_t + w = 0 \quad in \quad \mathbf{R}^n \times (0,\infty),\\ &g(x,0) = u_0(x), \quad x \in \mathbf{R}^n.\end{aligned}$$

Under the same hypotheses on w and w^- as in Lemma 2.1 and if in addition $w(x) = O(|x|^{-n-\epsilon})$ as $|x| \to \infty$ for some $\epsilon > 0$, there exist positive constants C and R_0 such that

$$g(x,t) \geq \frac{C}{|x|^{n-2}},$$

when $R_0 \leq |x| \leq 2R$ and $R^{4n} \leq t$.

Proof. Let h be the solution of the elliptic problem

$$\Delta h(x) + w(x) = 0 \quad x \in \mathbf{R}^n, \; h(x) \to 0 \text{ when } |x| \to \infty.$$

By Lemma 2.1, there exists $C_0 > 0$ such that

$$h(x) \geq C_0/|x|^{n-2}$$

when $|x|$ is sufficiently large. Since $|w(x)| \leq C/(1+|x|^{n+\epsilon})$ by assumption, there exists $C_1 > 0$ such that

$$h(x) \leq C_1/(1+|x|^{n-2})$$

for all $x \in \mathbf{R}^n$. Clearly $g - h$ is a solution of the following problem:

$$\begin{aligned} &\Delta(g-h) - (g-h)_t = 0 \quad (x,t) \in \mathbf{R}^n \times (0,\infty) \\ &g(x,0) - h(x) = -h(x). \end{aligned}$$

Suppose G is the fundamental solution of the heat equation. Then

$$g(x,t) - h(x) = -\int_{\mathbf{R}^n} G(x,t;y,0) h(y) dy.$$

Hence for $c_1 > 0$, $c_2 > 0$, and $t > 0$,

$$|g(x,t) - h(x)| \le \int_{\mathbf{R}^n} \frac{c_1}{t^{n/2}} e^{-c_2|x-y|^2/t} |h(y)| dy.$$

And

$$|g(x,t) - h(x)| \le \int_{\mathbf{R}^n} \frac{c_1 C_1}{t^{n/2}} e^{-c_2|x-y|^2/t} \frac{1}{1+|y|^{n-2}} dy,$$

which implies

$$\begin{aligned} |g(x,t) - h(x)| &\le \frac{C}{t^{1/4}} \int_{\mathbf{R}^n} \frac{|x-y|^{n-(1/2)}}{t^{(n/2)-(1/4)}} e^{-c_2|x-y|^2/t} \frac{1}{|x-y|^{n-(1/2)}|y|^{n-2}} dy \\ &\le \frac{C}{t^{1/4}} \int_{\mathbf{R}^n} \frac{1}{|x-y|^{n-(1/2)}(1+|y|^{n-2})} dy \\ &\le \frac{C}{t^{1/4}}, \end{aligned}$$

when $n \ge 3$. It follows, for large $|x|$, that

$$g(x,t) \ge \frac{C_0}{|x|^{n-2}} - \frac{C}{t^{1/4}}.$$

When $|x| \le 2R$ and $t \ge R^{4n}$, we have

$$g(x,t) \ge \frac{C}{|x|^{n-2}} - \frac{C}{R^n} \ge \frac{C}{2|x|^{n-2}},$$

when R is sufficiently large. ∎

Proof of Theorem 2.1: *Part* (a). We argue as in [Qi]. Let $\phi \in C^2(\mathbf{R}^n)$ be a positive function such that

(i) $\phi = 1$ in $B_1(0)$, $\phi = 0$ in $B_2^c(0)$ and $0 \le \phi \le 1$ everywhere.

(ii) $\partial\phi/\partial n = 0$ on $\partial(B_2(0) - B_1(0))$.

(iii) $|\Delta\phi/\phi^\alpha| \le C_\alpha$ in $B_2(0) - B_1(0)$ for all $\alpha \in (0,1)$.

For $0 < \alpha < 1$ such a ϕ exists. For example, let $\phi(x) = \exp(1 - 1/(1 - (1 - |x|)^4))$ in $B_2(0) - B_1(0)$ and extend it to all $\mathbf{R}^n$ in the usual manner.

Multiply Problem (1.1) by $\phi_R = \phi(x/R)$. Integrating by parts and using the properties of ϕ_R,

$$\frac{d}{dt}\int_{\mathbf{R}^n} \phi_R u dx - \int_{B_{2R}(0)-B_R(0)} u\Delta\phi_R dx = \int_{\mathbf{R}^n} |u|^p \phi_R dx + \int_{\mathbf{R}^n} w\phi_R dx. \quad (2.2)$$

By hypothesis, as $R \to +\infty$, $\int_{\mathbf{R}^n} w\phi_R dx \to \int_{\mathbf{R}^n} w dx > 0$. There exists $\delta > 0$ and $R_0 > 0$ such that $\int_{\mathbf{R}^n} w\phi_R dx \geq \delta$ when $R \geq R_0$.

From Property (iii) for ϕ, $|\Delta\phi_R|^q/\phi_R^{q-1} \leq CR^{-2q}$, and hence $\int_{B_{2R}(0)}(|\Delta\phi_R|^q/\phi_R^{q-1})dx \leq CR^{n-2q}$. From Hölder's inequality,

$$\int_{B_{2R}(0)-B_R(0)} u\Delta\phi_R dx \leq \left(\int_{\mathbf{R}^n} |u|^p \phi_R dx\right)^{1/p} \left(\int_{B_{2R}(0)} \frac{|\Delta\phi_R|^q}{\phi_R^{q-1}} dx\right)^{1/q},$$

where $1/p + 1/q = 1$.

Let, for $n \geq 1$, $F_R(t) = \int_{\mathbf{R}^n} u\phi_R dx$, $\overline{F_R}(t) = \int_{\mathbf{R}^n} |u|\phi_R dx$, and $G_R(t) = \int_{\mathbf{R}^n} |u|^p \phi_R dx$. Using these estimates in (2.2), we obtain

$$F_R'(t) \geq G_R(t) - CR^{n-2-(n/p)} G_R^{1/p}(t) + \delta \quad (2.3)$$

since $(n - 2q)/q = n - 2 - (n/p)$. The convex function $f(x) = x - CR^{n-2-(n/p)} x^{1/q}$ has a minimum value of $(1 - p)(p/C(R))^q$ where $C(R) = CR^{n-2-(n/p)}$ for $0 \leq x < +\infty$. Since $p < n/(n-2)$ here, and hence $n - 2 - (n/p) < 0$, it follows that $F_R' \geq \delta/2$ for all $t > 0$ and all sufficiently large R. It follows that $F_R(t) \geq \delta t/2 + F_R(0)$. Hence $F_R(t)$ and thus, by Hölder's inequality, $G_R(t)$ become arbitrarily large as t increases, regardless of the sign of $F_R(0)$. Consequently $F_R'(t) \geq CG_R(t) + \delta/2$ for all $t \geq t_0$. Thus, from Hölder's inequality, $\overline{F_R}(t) \geq F_R(t) \geq C_R \int_{t_0}^t \overline{F}_R^p(s) ds + C$. Let $g(t) = \int_{t_0}^t \overline{F_R}^p(s) ds$, then $g'(t) \geq [C_R g(t) + C]^p \geq Cg^p$. Thus g and hence u cannot exist for all time, i.e., $u \notin (C(0,T); \text{Ł}_{\text{loc}}^p)$. This proves part (a).

Proof of Theorem 2.1: (b) *The Critical Case,* $p = n/(n-2)$. Here we use Lemmas 2.1 and 2.2.

Let ϕ and ϕ_R be as in the proof of Theorem 2.1(a) and let $\eta(t) \in C^1$ be any function such that $\eta(t) = 0$ if $t \notin (1, 2)$, $|\eta'(t)/\eta^\alpha(t)| \leq C$ for all $\alpha \in (0, 1)$, and $t \in (1, 2)$. For example, $\eta(t) = \exp(1/((3 - 2t)^2 - 1))$ on $(1, 2)$.

For $R > 0$ and $T > 0$ define $Q = B_R(0) \times [T, 2T]$ and $\eta_T(t) = \eta(t/T)$.

Multiplying both sides of the differential equation in (1.1) by $\phi_R\eta_T$ and integrating over $\mathbf{R}^n$ we find

$$\begin{aligned} -\int_Q \eta_T'\phi_R u dx dt - \int_Q \eta_T u\Delta\phi_R dx dt \\ = \int_Q |u|^p \eta_T\phi_R dx dt + \int_Q w\eta_T\phi_R dx dt. \end{aligned} \quad (2.4)$$

Let

$$I(R,T) := \int_Q |u|^p \eta_T \phi_R dxdt.$$

Observe that since $p = n/(n-2)$,

$$\begin{aligned}\left|\int_Q \eta_T' \phi_R u dxdt\right| &\le I(R,T)^{1/p}\left(\int_Q \left(\frac{\eta_T'}{\eta^{1/p}}\right)^q \phi_R dxdt\right)^{1/q}\\ &= I(R,T)^{1/p}\gamma_0 R^2 T^{(2-n)/n},\\ \left|\int_Q \eta_T u \Delta\phi_R dxdt\right| &\le I(R,T)^{1/p}\left(\int_Q \frac{|\Delta\phi_R|^q}{\phi_R^{q-1}}\eta_T dxdt\right)^{1/q}\\ &= I(R,T)^{1/p}\gamma_1 T^{2/n}.\end{aligned}$$

Using these estimates in (2.4), there results

$$c_0 I^{1/p}(R,T)(R^{n/q}T^{(1-q)/q} + R^{(n-2q)/q}T^{1/q}) \ge I(R,T) + c_1\delta T$$

for positive constants c_0 and c_1 independent of R and T. We have used the assumption that $\int_{\mathbf{R}^n} w(x)dx > 0$ here. Noting that $q = n/2$, we have, for $T \ge R^2$, $c_2 I^{1/p}(R,T)T^{2/n} \ge I(R,T)$. From this, $I(R,T) \le c_3 T$. However, by Lemma 2.2, for $T = CR^{4n}$ with $C \ge 1$ and large R, $I(R,T) \ge \int_{5T/4}^{7T/4}\int_{R_0\le|x|\le R}(c/|x|^n)dxdt \ge c_4 T\ln T$ which shows that T cannot become arbitrarily large.

The statement of Part (c) was established in [Zh1] when $w > 0$ everywhere.

3. BOUNDARY VALUE PROBLEMS FOR EQUATIONS ON EXTERIOR DOMAINS

The next three theorems involve exterior boundary value problems in exterior domains. A key task is to identify a region where u is positive. We will state all the theorems first and then prove them.

Unless stated otherwise we make the following assumption on D.

Assumption 3.1. We assume that D has both the "exterior and interior rolling ball property," i.e., for any $x \in \partial D$ there exist spheres such that $B(x_0, r_0) \subset D \subset B(x_1, r_1)$, which are tangent to each other at x. Let R_0 be the largest of the r_0 and R_1 be the smallest r_1.

We shall also assume that $f \in L^\infty(\partial D)$. This assumption can sometimes be weakened.

THEOREM 3.1. *Let D satisfy Assumption* 3.1. *Suppose that $u_0(x) = f(x)$ for $x \in \partial D$. For $n \geq 3$, for Problem* (1.2),

(a) *If $p \leq n/(n-2)$ and $\int_{\partial D} f^+(x)dS > (R_1/R_0) \int_{\partial D} f^-(x)dS$, holds, then Problem* (1.2) *has no global solutions.*

(b) *If $p > n/(n-2)$, then Problem* (1.2) *has global solutions for all f with $\|f\|_{L^\infty(\partial D)}$ sufficiently small and all u_0 such that $|u_0(x)| \leq \epsilon/(1+|x|^{n+\delta})$ for any $\delta > 0$ and some sufficiently small $\epsilon > 0$.*

(c) *For any $p > 1$ and all $f < 0$, Problem* (1.2) *has nontrivial global solutions with $u_0 < 0$ and small in the sense of Part* (b) *above.*

Remark 3.1. It is not known whether or not the condition in Part (a) involving the shape of ∂D is necessary. It is worth noting that if ∂D is Lipschitz, our argument fails. When D is a ball, the condition becomes $\int_{\partial D} f(x)dS > 0$.

THEOREM 3.2. *For $n \geq 3$, the following is true for Problem* (1.3).

(a) *If $p \leq n/(n-2)$ and $\int_{\partial D} f(x)dx > 0$, then Problem* (1.3) *has no global solutions.*

(b) *If $p > n/(n-2)$, then Problem* (1.3) *has global solutions for all f with $\|f\|_{L^\infty(\partial D)}$ sufficiently small and all u_0 such that $|u_0(x)| \leq \epsilon/(1+|x|^{n+\delta})$ for any $\delta > 0$ and some sufficiently small $\epsilon > 0$.*

THEOREM 3.3. *Let D be as in Theorem* 3.1 *and $n \geq 3$. Suppose also* $\operatorname{dist}(\operatorname{supp}(w), D) > 0$. *For the problem*

$$\begin{aligned} &\Delta u + |u|^p - u_t + w = 0, && D^c \times (0,\infty),\\ &u(x,t) = 0, && x \in \partial D,\\ &u(x,0) = u_0(x), && x \in D^c. \end{aligned} \tag{1.1$'$}$$

we have:

(a) *If $p < n/(n-2)$, there exists $\lambda > 0$ such that $\int_{\mathbf{R}^n} w^+(x)dx > \lambda \int_{\mathbf{R}^n} w^-(x)dx$ and w^- satisfies condition* (2.1), *then* (1.1′) *has no global solutions. Here λ may depend on w and D.*

(b) *If $p > n/(n-2)$, then for any $\delta > 0$, there exists $\epsilon > 0$ such that Problem* (1.1) *has global solutions provided that $|w(x)|, |u_0(x)| \leq \epsilon/(1+|x|^{2+\delta})$ and $\int_{D^c} w(x)dx > 0$. There also exists u_0 for which the solution is not global for any w such that $\int_{D^c} w(x)dx > 0$.*

(b) *For any $p > 1$, there exists $w < 0$ such that* (1.1′) *has a global solution.*

Remark 3.2. The λ in Theorem 3.3 depends on the shape of ∂D and the support of w. At this time we do not have an explicit formula for it.

Next we need a lemma concerning solutions of Dirichlet problems for the Laplacian on exterior domains. The geometric assumptions on D made in Theorems 3.1–3.3 are necessary to guarantee the validity of Lemma 3.1.

LEMMA 3.1. (a) *Let D be as in Theorem* 3.1 *and $f \in L^1(\partial D)$. Suppose also that h is a solution of the problem*

$$\Delta h(x) = 0, \ x \in D^c; \qquad h(x) = f(x), \ x \in \partial D, \tag{3.1}$$

which is bounded on the exterior of a large sphere. If $\int_{\partial D} f^+(x) dS > (R_1/R_0) \int_{\partial D} f^-(x) dS$, then there exist positive constants c_1 and r_0 such that $h(x) \geq c_1/|x|^{n-2}$ when $|x| \geq r_0$.

(b) *Suppose h is a solution of the problem*

$$\begin{gathered} \Delta h(x) = 0, \ x \in D^c; \qquad \frac{\partial h(x)}{\partial n^+} = f(x), \ x \in \partial D, \\ h(x) \to 0, \ |x| \to \infty. \end{gathered} \tag{3.2}$$

If $\int_{\partial D} f^+(x) dS > \int_{\partial D} f^-(x) dS$, then there exist positive constants c_1 and r_0 such that $h(x) \geq c_1/|x|^{n-2}$ when $|x| \geq r_0$.

(c) *Let D be as in Theorem* 3.3. *Suppose h is a solution of the problem*

$$\Delta h(x) + w(x) = 0, \ x \in D^c; \qquad h(x) = 0, \ x \in \partial D. \tag{3.3}$$

If there is $\lambda > 0$ such that $\int_{D^c} w^+(x) dS > \lambda \int_{D^c} w^-(x) dS$ and if w^- satisfies the condition (2.1), *then there exist positive constants c_1 and r_0 such that $h(x) \geq c_1/|x|^{n-2}$ when $|x| \geq r_0$.*

Proof of the Lemma. Part (a). Given $y_0 \in \partial D$, suppose $B_{R_0}(x_0) \subset D \subset B_{R_1}(x_1)$ and that ∂D and $\partial B_{R_0}(x_0)$ are tangent to each other at y_0. Let P_0, P, and P_1 be the Poisson kernels of the complements of the above three regions respectively. By the maximum principle we have $P_1(x, y_0) \leq P(x, y_0) \leq P_0(x, y_0)$ when $x \in B^c_{R_1}(x_1)$. Hence for x sufficiently large

$$P(x, y_0) \leq \frac{-R_0^2 + |x - x_0|^2}{c_n R_0 |x - y_0|^n}.$$

This estimate implies

$$\limsup_{|x|\to\infty} |x|^{n-2} \int_{\partial D} P(x, y) f^-(y) dS_y \leq \frac{1}{c_n R_0} \int_{\partial D} f^-(y) dS_y.$$

Similarly,

$$\liminf_{|x|\to\infty} |x|^{n-2} \int_{\partial D} P(x, y) f^+(y) dS_y \geq \frac{1}{c_n R_1} \int_{\partial D} f^+(y) dS_y.$$

The first assertion now follows.

Part (b). Choosing $r > 0$ such that $D \subset B_r(0)$, we see that

$$\int_{\partial D} \frac{\partial h(y)}{\partial n} dS = \int_{|y|=r} \frac{\partial h(y)}{\partial n} dS.$$

Therefore we can assume that $D = B_r(0)$. Let $G = G(x, y)$ be the Neumann function for D^c. It is known that for $y \in \partial D$, $\lim_{|x|\to\infty} G(x, y)/|x|^{n-2} = c_n$. Therefore

$$\begin{aligned} h(x) &= \int_{\partial D} G(x, y) f(y) dS = \int_{\partial D} G(x, y) f^+(y) dS - \int_{\partial D} G(x, y) f^-(y) dS \\ &\geq \frac{c}{|x|^{n-2}} \left[\int_{\partial D} f^+(y) dS - (1+\epsilon) \int_{\partial D} f^+(y) dS \right]. \end{aligned}$$

Here ϵ goes to zero when $|x| \to \infty$. By assumption on f, there exists a $C > 0$ such that $h(x) \geq C/|x|^{n-2}$ when $|x|$ is large. This proves Part (b) of the lemma.

Part (c). We adopt the same notation as in part (a) of the lemma. For a chosen $y \in \operatorname{supp}(w)$, we can find balls $B_{R_0}(x_0)$, $B_{R_1}(x_1)$ which are tangent to ∂D at $y_0 \in \partial D$. Moreover, $y \in B^c_{R_1}(x_1)$. Let G_0, G, and G_1 be the Dirichlet Green's functions for $B_{R_0}(x_0)$, D^c and $B_{R_1}(x_1)$ respectively. By the maximum principle, for x sufficiently large $G_1(x, y) \leq G(x, y) \leq G_0(x, y)$. For simplicity we take $x_0 = 0$, then

$$G_0(x, y) = c_n \left(\frac{1}{|x-y|^{n-2}} - \frac{R_0^{n-2}}{|y|^{n-2} |x - (R_0^2 y/|y|^2)|^{n-2}} \right).$$

Using the assumption that $\operatorname{dist}(y, D) > 0$, we can find positive constants C_1 and C_2 such that $C_1 \leq \lim_{|x|\to\infty} G(x, y)|x|^{n-2} \leq C_2$ when $y \in \operatorname{supp}(w) \cap B_R(0)$ and $|x|$ is sufficiently large. After writing

$$\begin{aligned} h(x) &\geq \int_{B_R(0)-D} G(x, y) w^+(y) dy - \int_{B_R(0)-D} G(x, y) w^-(y) dy \\ &\quad - \int_{B_R^c(0)} G(x, y) w^-(y) dy \\ &\geq \frac{C_1}{|x|^{n-2}} \int_{B_R(0)-D} w^+(y) dy - \frac{C_2}{|x|^{n-2}} \int_{B_R(0)-D} w^+(y) dy \\ &\quad - \int_{B_R^c(0)} G(x, y) w^-(y) dy, \end{aligned}$$

we can use an argument similar to Part (a) to conclude $h(x) \geq c/|x|^{n-2}$ when $|x|$ is large. Note that λ depends on C_1 and C_2 and hence on the shape of D. ∎

LEMMA 3.2. (a) *Let $g = g(x,t)$ be the solution of the linear problem*

$$\begin{array}{lll} \Delta g - g_t = 0 & in & D^c \times (0,\infty), \\ g(x,t) = f(x) & in & \partial D \times (0,\infty), \\ g(x,0) = u_0(x) & & x \in D^c. \end{array} \tag{3.1$'$}$$

Let D be as in Lemma 3.1(a) and suppose that $\int_{\partial D} f^+(x)dS > (R_1/R_0)\times \int_{\partial D} f^-(x)dS$. Then there exist positive constants C and R_0 such that $g(x,t) \geq C/R^{n-2}$, when $R_0 \leq R \leq |x| \leq 2R$ and $R^{4n} \leq t$.

(b) *Let $g = g(x,t)$ be the solution of the linear problem*

$$\begin{array}{lll} \Delta g - g_t = 0 & in & D^c \times (0,\infty), \\ \dfrac{\partial g(x,t)}{\partial n^+} = f(x) & in & \partial D,\ g(x,t) \to 0,\ |x| \to \infty. \times (0,\infty), \\ g(x,0) = u_0(x), & & x \in D^c. \end{array} \tag{3.2$'$}$$

Suppose $\int_{\partial D} f^+(x)dS > \int_{\partial D} f^-(x)dS$. Then there exist positive constants C and R_0 such that $g(x,t) \geq C/R^{n-2}$, when $R_0 \leq R \leq |x| \leq 2R$ and $R^{4n} \leq t$.

(c) *Let $g = g(x,t)$ be the solution of the linear problem*

$$\begin{array}{lll} \Delta g - g_t + w = 0 & in & D^c \times (0,\infty), \\ g(x,t) = 0 & in & \partial D \times (0,\infty), \\ g(x,0) = u_0(x), & & x \in D^c. \end{array} \tag{3.3$'$}$$

Let D be as in Lemma 3.1(c) and suppose there exists $\lambda > 0$ such that $\int_{D^c} w^+(x)dS > \lambda \int_{D^c} w^-(x)dS$ and w^- satisfies condition (2.1). Then there exist positive constants C and R_0 such that $g(x,t) \geq C/R^{n-2}$, when $R_0 \leq R \leq |x| \leq 2R$ and $R^{4n} \leq t$.

Proof. We only prove part (a) as parts (b) and (c) follow from Lemma 3.1(b) and (c) respectively using Gaussian upper bounds for Neumann kernels (see [CWZ]). Let h be the solution of the exterior elliptic problem

$$\begin{array}{ll} \Delta h(x) = 0, & x \in D^c \\ h(x) = f(x), & x \in \partial D. \end{array} \tag{3.4}$$

Assume that $0 \in D$. By hypothesis on f and Lemma 3.1, there exists $C_0 > 0$ such that

$$h(x) \geq C_0/|x|^{n-2} \tag{3.5}$$

when $|x|$ is sufficiently large. Since $f \in L^1(\partial D)$, there exists $C_1 > 0$ such that

$$h(x) \leq C_1/|x|^{n-2} \tag{3.6}$$

for all $x \in D^c$. Clearly $g - h$ is a solution of the following problem:

$$\begin{aligned} &\Delta(g-h) - (g-h)_t = 0, && (x,t) \in D^c \times (0,\infty) \\ &g(x,t) - h(x) = 0, && x \in \partial D, \\ &g(x,0) - h(x) = u_0(x) - h(x). && \end{aligned} \tag{3.7}$$

Suppose G_{D^c} is the Green's function corresponding to the initial-boundary value problem for the heat equation in $D^c \times (0,\infty)$ vanishing on $\partial D^c \times (0,\infty)$. Then

$$g(x,t) - h(x) = \int_{D^c} G_{D^c}(x,t;y,0)(u_0(y) - h(y))dy. \tag{3.8}$$

From the standard theory of parabolic equations, for $c_1 > 0$, $c_2 > 0$, and $t > 0$,

$$G_{D^c}(x,t;y,0) \le \frac{c_1}{t^{n/2}} e^{-c_2|x-y|^2/t}. \tag{3.9}$$

Hence

$$|g(x,t) - h(x)| \le \frac{c_1}{t^{n/2}} \int_{D^c} e^{-c_2|x-y|^2/t}(|u_0(y)| + |h(y)|)dy + c_2\frac{1}{t^{n/2}}. \tag{3.10}$$

By (3.6),

$$|g(x,t) - h(x)| \le \frac{c_1 C_1}{t^{n/2}} \int_{D^c} e^{-c_2|x-y|^2/t} \frac{1}{|y|^{n-2}} dy + c_2 \frac{1}{t^{n/2}}, \tag{3.11}$$

which implies

$$\begin{aligned} |g(x,t) - h(x)| &\le \frac{C}{t^{1/4}} \int_{D^c} \frac{|x-y|^{n-(1/2)}}{t^{(n/2)-(1/4)}} e^{-c_2|x-y|^2/t} \\ &\qquad \times \frac{1}{|x-y|^{n-(1/2)}|y|^{n-2}} dy + c_2 \frac{1}{t^{n/2}} \\ &\le \frac{C}{t^{1/4}} \int_{D^c} \frac{1}{|x-y|^{n-(1/2)}|y|^{n-2}} dy + c_2 \frac{1}{t^{n/2}} \\ &\le \frac{C}{t^{1/4}}, \end{aligned} \tag{3.12}$$

when $n \ge 3$. The inequality follows since 0 is in the interior of D. By (2.3) and (2.9), when $|x|$ is large,

$$g(x,t) \ge \frac{C_0}{|x|^{n-2}} - \frac{C}{t^{1/4}}. \tag{3.13}$$

When $R \le |x| \le 2R$ and $t \ge R^{4n}$, $g(x,t) \ge C/R^{n-2} - C/R^n \ge C/2R^{n-2}$, when R is sufficiently large. ∎

Proof of Theorem 3.1 (a). Let $\phi, \eta \in C^\infty[0,\infty)$ be two functions satisfying

(i) $0 \le \phi \le 1$; $\phi(r) = 1$, $r \in [2,3]$; $\phi(r) = 0$, $r \in [0,1) \cup (4,\infty)$;

(ii) $\phi'(1) = \phi'(4) = 0$, $|\phi'(r)| \le C$, $|\phi''(r)| \le C$;

(iii) $0 \le \eta \le 1$; $\eta(t) = 1$, $t \in [0, 1/4]$; $\eta(t) = 0$, $t \in [1,\infty)$; $-C \le \eta'(t) \le 0$.

For $R > 0$ define $Q_R = (B_{4R}(0) - B_R(0)) \times [R^{4n}, R^{4n} + R^2]$. Let $\psi_R = \phi_R(|x|)\eta_R(t)$, where $\phi_R(r) = \phi(r/R)$ and $\eta_R = \eta((t - R^{4n})/R^2)$, be our cut off function. Clearly,

$$\left|\frac{\partial \phi_R}{\partial r}\right| \le \frac{C}{R}; \qquad \left|\frac{\partial^2 \phi_R}{\partial r^2}\right| \le \frac{C}{R^2}; \qquad -\frac{C}{R^2} \le \eta_R'(t) \le 0, \tag{3.14}$$

$$\frac{\partial \phi_R(x)}{\partial r} = 0, \tag{3.15}$$

when $|x| = R$ or $|x| = 4R$.

We derive a contradiction. Let u be a global solution of Problem (1.2) and $g = g(x,t)$ be the solution of the linear problem

$$\begin{aligned} \Delta g - g_t &= 0 \text{ in } D^c \times (0,\infty), \\ g(x,t) &= f(x) \text{ in } \partial D \times (0,\infty), \\ g(x,0) &= u_0(x), \qquad x \in D^c. \end{aligned} \tag{3.16}$$

By the maximum principle, $u \ge g$.

Since D is bounded, we select $R > 0$ so that $D \subset B_R(0)$. For such $R > 0$ we set

$$I_R \equiv \int_{Q_R} |u|^p(x,t)\psi_R^q(x,t)\,dxdt, \tag{3.17}$$

where $1/p + 1/q = 1$. By Lemma 3.2, when R is large $u(x,t) \ge g(x,t) \ge c/|x|^{n-2}$ in the cube Q_R. Therefore $I_R \ge \int_{Q_R} g^p\psi_R^q(x,t)\,dxdt$. Multiplying both sides of the differential equation in (1.2) by ψ_R and integrating,

$$I_R + \int_{Q_R} g^p\psi_R^q(x,t)\,dxdt \le 2\int_{Q_R} [u_t(x,t) - \Delta u(x,t)]\psi_R^q(x,t)\,dxdt, \tag{3.18}$$

which implies, via integration by parts,

$$\begin{aligned} &\frac{1}{2}I_R + \frac{1}{2}\int_{Q_R} g^p\psi_R^q(x,t)\,dxdt \\ &\quad \le \int_{B_{4R}(0)} u(x,.)\psi_R^q(x,.)\big|_{R^{4n}}^{R^{4n}+R^2}\,dx \end{aligned}$$

$$
\begin{aligned}
&-\int_{Q_R} u(x,t)\phi_R^q(x) q\eta_R^{q-1}(t)\eta_R'(t)\,dxdt \\
&+\int_{R^{4n}}^{R^{4n}+R^2}\int_{\partial B_{4R}(0)} u(x,t)\frac{\partial\phi_R^q(x)}{\partial n}\eta_R^q(t)\,dS_x dt \\
&-\int_{R^{4n}}^{R^{4n}+R^2}\int_{\partial B_{4R}(0)} \psi_R^q\frac{\partial u}{\partial n}(x,t)\,dS_x dt \\
&-\int_{R^{4n}}^{R^{4n}+R^2}\int_{\partial B_{R}(0)} u(x,t)\frac{\partial\phi_R^q(x)}{\partial n}\eta_R^q(t)\,dS_x dt \\
&+\int_{R^{4n}}^{R^{4n}+R^2}\int_{\partial B_{R}(0)} \psi_R^q\frac{\partial u}{\partial n}(x,t)\,dS_x dt \\
&-\int_{Q_R} u(x,t)\Delta\phi_R^q(x)\eta_R^q(t)\,dxdt. \qquad (3.19)
\end{aligned}
$$

Noting that $u(x, R^{4n}) \geq 0$, $\psi_R(x, R^{4n}+R^2) = 0$, $\frac{\partial\phi_R^q}{\partial n} = q\phi_R^{q-1}\frac{\partial\phi_R}{\partial r}\frac{\partial r}{\partial n} = 0$ and $\psi_R(x,t) = 0$ on the lateral boundary of Q_R, we obtain

$$
\begin{aligned}
I_R &+ 2\int_{Q_R} g^p\psi_R^q(x,t)\,dxdt \\
&\leq -2q\int_{Q_R} u(x,t)\phi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)\,dxdt \\
&\quad -\int_{Q_R} u(x,t)\Delta\phi_R^q(x)\eta_R^q(t)\,dxdt. \qquad (3.20)
\end{aligned}
$$

Since $\Delta\phi_R^q = q\phi^{q-1}\Delta\phi_R + q(q-1)\phi_R^{q-2}|\nabla\phi_R|^2$, (3.20) yields

$$
\begin{aligned}
I_R &+ \int_{Q_R} g^p\psi_R^q(x,t)\,dxdt \\
&\leq -2q\int_{Q_R} u(x,t)\phi_R^q(x)\eta_R^{q-1}(t)\eta_R(t)\,dxdt \\
&\quad -2\int_{Q_R} u(x,t)q(\phi_R^{q-1}\Delta\phi_R)(x)\eta_R^q(t)\,dxdt. \qquad (3.21)
\end{aligned}
$$

Noting the supports of ϕ_R and η_R and using Lemma 3.2, for large R,

$$
\begin{aligned}
\int_{Q_R} g^p\psi_R^q(x,t)\,dxdt &\geq \int_{R^{4n}}^{R^{4n}+(R^2/4)}\int_{B_{3R}(0)-B_{2R}(0)} g^p(x,t)\,dxdt \\
&\geq \frac{CR^{n+2}}{R^{(n-2)p}} \geq CR^2
\end{aligned}
$$

since $p < n/(n-2)$. Therefore one can reduce (3.21) to

$$I_R + C_0R^2 \le -2q\int_{R^{4n}}^{R^{4n}+R^2}\int_{B_{4R}(0)} u(x,t)\phi_R^q(x)\eta_R^{q-1}(t)\eta_R'(t)dxdt$$
$$-2q\int_{R^{4n}}^{R^{4n}+R^2}\int_{B_{4R}(0)-B_R(0)} u(x,t)(\phi_R^{q-1}\Delta\phi_R)(x)\eta_R^q(t)dxdt, \tag{3.22}$$

where C_0 is a positive-number-independent R which is sufficiently large.

Since ϕ_R is radial, $\Delta\phi_R = \phi_R'' + ((n-1)/r)\phi_R'$. Taking R sufficiently large,

$$|\Delta\phi_R| \le \frac{C}{R^2}, \tag{3.23}$$

when $x \in B_{4R}(0) - B_R(0)$. Using the estimate (3.23) in (3.22),

$$I_R + C_0R^2 \le C\int_{R^{4n}}^{R^{4n}+R^2}\int_{B_{4R}(0)} u(x,t)\phi_R^q(x)\eta_R^{q-1}(t)\frac{1}{R^2}dxdt$$
$$+ C\int_{R^{4n}}^{R^{4n}+R^2}\int_{B_{4R}(0)-B_R(0)} u(x,t)\phi_R^{q-1}\eta_R^q(t)\frac{1}{R^2}dxdt.$$

Since $\phi_R, \eta_R \le 1$, by Hölder's inequality

$$I_R + C_0R^2 \le \frac{C}{R^2}\left(\int_{R^{4n}}^{R^{4n}+R^2}\int_{B_{4R}(0)} u^p\psi_R^{p(q-1)}(x,t)dxdt\right)^{1/p}$$
$$\times\left(\int_{R^{4n}}^{R^{4n}+R^2}\int_{B_{4R}(0)} dxdt\right)^{1/q}$$
$$+\frac{C}{R^2}\left(\int_{R^{4n}}^{R^{4n}+R^2}\int_{B_{4R}(0)-B_R(0)} u^p\psi_R^{p(q-1)}(x,t)dxdt\right)^{1/p}$$
$$\times\left(\int_{R^{4n}}^{R^{4n}+R^2}\int_{B_{4R}(0)-B_R(0)} dxdt\right)^{1/q}. \tag{3.24}$$

Therefore

$$I_R + C_0R^2 \le C\left(\int_{R^{4n}}^{R^{4n}+R^2}\int_{B_{4R}(0)} u^p(x,t)\psi_R^q(x,t)dxdt\right)^{1/p} R^{(n+2)/q-2}$$
$$+C\left(\int_{R^{4n}}^{R^{4n}+R^2}\int_{B_{4R}(0)-B_R(0)} u^p(x,t)\psi_R^q(x,t)dxdt\right)^{1/p} R^{(n+2)/q-2}, \tag{3.25}$$

which yields

$$I_R + C_0R^2 \le C_1 I_R^{1/p} R^{(n+2)/q-2}. \tag{3.26}$$

Thus $I_R \le C_1^q R^{n+2-2q} \equiv C_4 R^{n+2-2q}$. Again from (3.26), $C_0 R^2 \le C_1 I_R^{1/p} R^{(n+2)/q-2}$ or $I_R \ge (C_0/C_1)^p R^{n+2-(n-2)p}$. Combining these two inequalities, we must have $R^{n+2-(n-2)p-(n+2-2q)} = R^{2q-(n-2)p} \le$ const. But $2q-(n-2)p = [n-(n-2)p]/(p-1) > 0$. Thus R must be bounded, an obvious contradiction. In the critical case we have $I_R = O(R^2)$ for large R. The discussion is the same as for Theorem 2.1(b) and will therefore be omitted.

Proof of (b). As remarked earlier the existence part follows from standard arguments.

Proof of (c). We solve the exterior parabolic problem

$$\begin{array}{ll} Lu = \Delta u(x,t) + |u|^p(x,t) - u_t(x,t) = 0, & (x,t) \in D^c \times [0,\infty) \\ u(x,t) = f(x) < 0, & (x,t) \in \partial D \times [0,\infty), \\ u(x,0) = 0, & x \in D^c. \end{array}$$

Since $f < 0$, $u = 0$ is a super solution. Take $C >> 1$ then $L(-C) = |-C|^p > 0$, so $-C$ is a subsolution. Thus the above exterior problem has a global solution. ∎

Proof of Theorem 3.2. (a). Let R be so large that $D \subset B_R(0)$. Define $Q_R = (B_{2R}(0) - D) \times [R^{4n}, R^{4n} + R^2]$. We again use a cut-off function: $\psi_R = \phi_{D,R}(x)\eta_R(t)$, where $\eta_R = \eta((t - R^{4n})/R^2)$ for η defined at the beginning of Section 3. $\phi_{D,R} \in C_0^\infty(D^c)$ is such that $\phi_{D,R}(x) = 1$ when $x \in B_R(0) - D$; $\phi_{D,R}(x) = 0$ when $x \in B_{2R}(0)^c$; $0 \le \phi_{D,R}(x) \le 1$ when $x \in B_{2R}(0) - B_R(0)$. We also choose $\phi_{D,R}(x)$ to be radial in the ring $B_{2R}(0) - B_R(0)$ and to satisfy

$$\left|\frac{\partial \phi_{D,R}}{\partial r}\right| \le \frac{C}{R}; \qquad \left|\frac{\partial^2 \phi_{D,R}}{\partial r^2}\right| \le \frac{C}{R^2}; \qquad -\frac{C}{R^2} \le \eta_R'(t) \le 0, \tag{3.27}$$

$$\frac{\partial \phi_{D,R}(x)}{\partial r} = 0, \tag{3.28}$$

when $|x| = R$ or $|x| = 2R$.

We seek a contradiction. Let u be a global positive solution of Problem (1.3). Set

$$I_R \equiv \int_{Q_R} |u|^p(x,t)\psi_R^q(x,t)\,dxdt, \tag{3.29}$$

where $1/p + 1/q = 1$. From (1.3),

$$I_R = \int_{Q_R} [u_t(x,t) - \Delta u(x,t)]\psi_R^q(x,t)\,dxdt. \tag{3.30}$$

After integration by parts,

$$
\begin{aligned}
I_R = & \int_{B_{2R}(0)-D} u(x,.)\psi_R^q(x,.)|_{R^{4n}}^{R^{4n}+R^2}\,dx \\
& - \int_{Q_R} u(x,t)\phi_{D,R}^q(x) q\eta_R^{q-1}(t)\eta_R'(t)\,dx\,dt \\
& + \int_{R^{4n}}^{R^{4n}+R^2} \int_{\partial B_{2R}(0)} u(x,t)\frac{\partial\phi_{D,R}^q(x)}{\partial n}\eta_R^q(t)\,dS_x\,dt \\
& - \int_{R^{4n}}^{R^{4n}+R^2} \int_{\partial B_{2R}(0)} \psi_R^q\frac{\partial u}{\partial n}(x,t)\,dS_x\,dt \\
& - \int_{R^{4n}}^{R^{4n}+R^2} \int_{\partial D} u(x,t)\frac{\partial\phi_{D,R}^q(x)}{\partial n}\eta_R^q(t)\,dS_x\,dt \\
& + \int_{R^{4n}}^{R^{4n}+R^2} \int_{\partial D} \psi_R^q\frac{\partial u}{\partial n}(x,t)\,dS_x\,dt \\
& - \int_{Q_R} u(x,t)\Delta\phi_{D,R}^q(x)\eta_R^q(t)\,dx\,dt.
\end{aligned}
\tag{3.31}
$$

Then, from the definitions of $\phi_{R,D}$ and η_R,

$$
\begin{aligned}
I_R \le & \int_{R^{4n}}^{R^{4n}+R^2} \int_{\partial D} \psi_R^q\frac{\partial u}{\partial n}(x,t)\,dS_x\,dt \\
& - \int_{Q_R} u(x,t)\phi_{D,R}^q(x) q\eta_R^{q-1}(t)\eta_R'(t)\,dx\,dt \\
& - \int_{Q_R} u(x,t)\Delta\phi_{D,R}^q(x)\eta_R^q(t)\,dx\,dt.
\end{aligned}
\tag{3.32}
$$

Since $\Delta\phi_{D,R}^q = q\phi_{D,R}^{q-1}\Delta\phi_{D,R} + q(q-1)\phi_{D,R}^{q-2}|\nabla\phi_{D,R}|^2$, we have, since $\partial u/\partial n = -\partial u/\partial n^+$,

$$
\begin{aligned}
& I_R + \int_{R^{4n}}^{R^{4n}+R^2} \int_{\partial D} \psi_R^q\frac{\partial u}{\partial n^+}(x,t)\,dS_x\,dt \\
& \quad \le -\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_R(0)} u(x,t)\phi_{D,R}^q(x) q\eta_R^{q-1}(t)\eta_R'(t)\,dx\,dt \\
& \qquad - \int_{R^{4n}}^{R^{4n}+R^2} \int_{B_R(0)-B_{R/2}(0)} u(x,t)q(\phi_R^{q-1}\Delta\phi_{D,R})(x)\eta_R^q(t)\,dx\,dt.
\end{aligned}
\tag{3.33}
$$

By hypothesis, $\int_{\partial D} f(x)dS > 0$, so that $\int_{R^{4n}}^{R^{4n}+R^2} \int_{\partial D} \psi_R^q(\partial u/\partial n^+)(x,t)dS_x\, dt \geq C_0R^2$ for some $C_0 > 0$. Hence

$$I_R + C_0R^2 \leq -\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{2R}(0)-D} u(x,t)\phi_{D,R}^q(x)q\eta_R^{q-1}(t)\eta_R'(t)dxdt$$
$$-\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{2R}(0)-D} u(x,t)q(\phi_R^{q-1}\Delta\phi_{D,R})(x)\eta_R^q(t)dxdt. \tag{3.34}$$

The same arguments as in the proof of Theorem 3.1(a) yield (3.26) for this I_R. The remainder of the argument is the same as that following equation (3.26). In both cases, we conclude, since R cannot be bounded, that $I_R = +\infty$.

Part (b) can be proved by a fixed point argument as in [Zh1]. ∎

Proof of Theorem 3.3. Let Q_R, ψ_R and I_R be as in Theorem 3.1, i.e., $Q_R = (B_{4R}(0) - B_R(0)) \times [R^{4n}, R^{4n} + R^2]$ and $I_R = \int_{Q_R} |u|^p\psi_R^q(x,t)dxdt$. Under the assumption of Theorem 3.3, one can follow the proof of Theorem 3.1(a) to obtain

$$I_R + \int_{Q_R} w\psi_R^q dxdt \leq CI_R^{1/p}R^{(n+2)/q-2}. \tag{3.35}$$

Since w^- satisfies (2.1), we know for $x \in B_{4R}(0) - B_R(0)$ that

$$\frac{c}{R^{n-2}}\int_{B_{4R}(0)-B_R(0)} w^-(y)dy \leq C\int_{B_{4R}(0)-B_R(0)} \frac{w^-(y)}{|x-y|^{n-2}}dy \leq \frac{0(1)}{R^{n-2}}.$$

Therefore

$$\int_{Q_R} w^-\psi_R^q dxdt = 0(1)R^2. \tag{3.36}$$

Let g be the solution of the problem

$$\begin{array}{ll} \Delta g - g_t + w = 0, & \text{in } D^c \times (0,\infty), \\ g(x,t) = 0, & \text{in } \partial D^c \times (0,\infty), \\ g(x,0) = 0, & \text{in } D^c. \end{array}$$

By the Maximum Principle and Lemma 3.2(c), $u(x,t) \geq g(x,t) \geq C/R^{n-2}$ in the region Q_R. Therefore

$$I_R \geq \int_{R^{4n}}^{R^{4n}+R^2/4} \int_{B_{3R}(0)-B_{2R}(0)} |u|^p dxdt \geq R^{n+2-p(n-2)}. \tag{3.37}$$

On the other hand from (3.37) and (3.36) and since $p < n/(n-2)$, it is clear that $I_R \gg \int_{Q_R} w\psi_R^q dxdt$. Hence from (3.37), $I_R \leq CR^{n+2-2q}$, when $p < n/(n-2)$. This contradicts (3.39) when R is large since $n+2-2q < n+2-p(n-2)$. This proves Part (a). Part (b) of the theorem again follows from standard arguments. Part (c) follows from an easy super- and subsolution argument as in Theorem 3.1. ∎

4. BLOW UP AND EXISTENCE RESULTS FOR A SYSTEM

In this section we extend the results obtained for single equations to a system of weakly coupled equations. The strategy in proving the blow-up result remains the same i.e., one first identifies a region where solutions are positive. Then one tries to show that certain integrals of the solution on some region will blow up in finite time. The computation, though more complicated, is an amalgemation of those in the previous sections and those in [Zh2]. Therefore we will refer to that paper quite often.

We consider the following Cauchy problem:

$$\begin{array}{lll} \Delta u - \partial_t u + |v|^p + w_1 = 0 & \text{in} & \mathbf{R}^n \times (0,\infty), \\ \Delta v - \partial_t v + |u|^q + w_2 = 0 & \text{in} & \mathbf{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) \quad v(x,0) = v_0(x) & \text{in} & \mathbf{R}^n, \end{array} \tag{4.1}$$

The assumptions imposed on w_1 and w_2 are similar to those for w in Section 2. To prove global nonexistence, we assume that for $i = 1, 2$,

$$\int_{\mathbf{R}^n} w_i(x)dx > 0, \qquad \int_{B_R^c(0)} \frac{w_i^-(y)}{|x-y|^{n-2}}dy = \frac{o(1)}{|x|^{n-2}} \quad \text{for large } R. \tag{4.2}$$

THEOREM 4.1. (a) *When $p \geq q > 1$ and $p(q+1)/(pq-1) > n/2$, Problem* (4.1) *possesses no global solutions if both w_1 and w_2 satisfy* (4.2).

(b) *When $p \geq q > 1$ and $p(q+1)/(pq-1) = n/2$, Problem* (4.1) *possesses no global solutions if either of w_1 or w_2 satisfies* (4.2) *and $u_0, v_0 \geq 0$.*

(c) *Suppose $p \geq q > 0$ and $p(q+1)/(pq-1) < n/2$. Then Problem* (4.1) *has global positive solutions whenever $w_1(x), w_2(x), u_0(x), v_0(x)$ are all nonnegative and are bounded above by $\epsilon/(1+|x|^{n+\delta})$ for some $\delta > 0$ and some sufficiently small $\epsilon > 0$.*

Remark 4.1. If one is only concerned with nonnegative solutions, then a result similar to Theorem 4.1(a) can be proved by the known method in [L2] or by slightly modifying the proof in [Z2].

The proof of this is rather similar to that of Theorem 4.1 remodeled along the lines of the corresponding result in Section 3 for single equations and is therefore omitted.

Proof of Theorem 4.1. Part (a). Let Q_R and ψ_R be as in Section 2. Let (u, v) be a global positive solution of (1.1). For $R > 0$ we set

$$I_R \equiv \int_{Q_R} |v|^p(x,t)\psi_R^{q'}(x,t)dxdt \tag{4.3}$$

$$J_R \equiv \int_{Q_R} |u|^q(x,t)\psi_R^{q'}(x,t)dxdt, \tag{4.4}$$

where $\frac{1}{q}+\frac{1}{q'}=1$. Since (u, v) is a solution of (4.1),

$$I_R = \int_{Q_R} [u_t(x,t) - \Delta u(x,t) - w_1(x,t)]\psi_R^{q'}(x,t)dxdt.$$

From Lemma 2.2, $u(x,t) \geq 0$ when $(x,t) \in Q_R$ and R is large. From Section 2, $\int_{Q_R} w_1 \psi_R^{q'} \geq 0$ when R is large. As in [Zh2], we obtain

$$I_R \leq C J_R^{1/q} R^{(n+2)/q'-2}. \tag{4.5}$$

$$J_R \leq C I_R^{1/p} R^{(n+2)/p'-2}. \tag{4.6}$$

Substituting (4.6) into (4.5), one obtains

$$I_R \leq C R^{(pq/((pq-1))[((n+2)/p'-2)\left(\frac{1}{q}\right)+(n+2)/q'-2]}. \tag{4.7}$$

From Lemma 2.2 again, $v(x,t) \geq C/R^{n-2}$ when $(x,t) \in Q_R$ and R is large. Hence

$$I_R \geq C R^{(n+2)-p(n-2)}. \tag{4.8}$$

It is clear that

$$\frac{pq}{pq-1}\left[\left(\frac{n+2}{p'}-2\right)\frac{1}{q}+\frac{n+2}{q'}-2\right] = \frac{(n+2)(pq-1)-2p-2pq}{pq-1}$$
$$= n+2-2\frac{p(q+1)}{pq-1}. \tag{4.9}$$

Switching the roles of p and q

$$J_R \leq C R^{(pq/(pq-1))[((n+2)/q'-2)(1/p)+(n+2)/p'-2]} \leq C R^{n+2-2q(p+1)/(pq-1)}, \tag{4.10}$$

$$J_R \geq C R^{n+2-q(n-2)}. \tag{4.11}$$

Thus,

$$R^{2q(p+1)/(pq+1)-q(n-2)} \leq \text{constant}. \tag{4.12}$$

The exponent on R will be positive if $(p+1)/(pq-1) > n/2-1$. However, $(p+1)/(pq-1)+1 = p(q+1)/(pq-1) > n/2$ by hypothesis. Thus R is bounded, an impossibility.

Proof of part (b). Here $p(q+1)/(pq-1) = n/2$. From (4.7) and (4.9)

$$I_R = \int_{Q_R} |v|^p(x,t)\psi_R^{q'}(x,t)dxdt \leq CR^2. \tag{4.13}$$

Equation (4.13) will lead to a contradiction. By (4.13) and the shape of Q_R,

$$\int_{R^{4n}+R^2/2}^{R^{4n}+3R^2/2} \int_{B_{R/2}(x_0)} |v|^p(x,t)dxdt \leq I_R \leq CR^2, \tag{4.14}$$

for all large $R > 0$. From (4.14) and the Mean Value Theorem,

$$\inf_{R^{4n} \le t \le 2R^{4n}} \int_{B_{R/2}(x_0)} |v|^p(x, t)dx \le C.$$

Hence there exist sequences $\{R_j\}$ and $t_j \in [R_j^{4n}, 2R_j^{4n}]$, such that $\lim_{j\to\infty} R_j = \infty$, and

$$\int_{B_{R_j}(0)} |v|^p(x, t_j)dx \le C. \tag{4.15}$$

From the Maximum Principle and the assumption that $u_0, v_0 \ge 0$,

$$u(x, t) \ge \int_0^t \int_{\mathbf{R}^n} G(x, t; y, s)w_1(y)dyds. \tag{4.16}$$

Given any $x \in B_{R_j}(0) - B_{R_1}(0)$ and $t \ge R_j^{4n}$ with j and R_1 large, from Lemma 2.2 it follows that $\int_0^t \int_{\mathbf{R}^n} G(x, t; y, s)w_2(y, s)dyds \ge 0$. Fixing large $R > 0$, for $c \le |y| \le R$ and $s \ge R^{4n}$, by Lemma 2.2 again,

$$u(y, s) \ge \int_0^y \int_{\mathbf{R}^n} G(y, s; z, \tau)w_1(z)dzd\tau \ge C/|y|^{n-2}.$$

Hence

$$\begin{aligned} v(x, t) &\ge \int_0^t \int_{\mathbf{M}^n} G(x, t; y, s)|u|^q(y, s)dyds \\ &\ge \int_{R^{4n}}^t \int_{c\le|y|\le R} G(x, t; y, s)\left[\int_0^s \int_{\mathbf{R}^n} G(y, t; z, \tau)w_1(z)dzd\tau\right]^q dyds \\ &\ge \int_{R^{4n}}^t \int_{c\le|y|\le R} G(x, t; y, s)\frac{C}{|y|^{q(n-2)}}dyds \\ &= \int_{c\le|y|\le R} \frac{C}{|y|^{q(n-2)}} \int_0^{t-R^{4n}} G(x, s; y, 0)dsdy. \end{aligned}$$

Therefore

$$F(x) \equiv \liminf_{t\to\infty} v(x, t) \ge \int_{c\le|y|\le R} \frac{C}{|y|^{q(n-2)}|x-y|^{n-2}}dy.$$

Since R can be arbitrarily large

$$F(x) \ge \int_{c\le|y|} \frac{C}{|y|^{q(n-2)}|x-y|^{n-2}}dy \ge \frac{C}{|x|^{q(n-2)-2}}. \tag{4.17}$$

By (4.15), (4.17), and Fatou's Lemma, we can find an $R_1 > 0$ such that

$$\int_{B_R(x_0)-B_{R_1}(x_0)} \frac{1}{|x|^{p[q(n-2)-2]}}dx \le C \tag{4.18}$$

when $R > R_1$. Since we are in the critical case, $p[q(n-2)-2] = n$ and hence $\int_{B_R(0)-B_{R_1}(0)}(dx/|x|^n) \le C$. This leads to a contradiction since the left-hand side becomes infinite as $R \to \infty$. Thus global solutions to (4.1) do not exist. This proves part (b). Part (c) is just Theorem 1.1 in [Zh2]. ∎

5. EXTENSIONS AND OPEN PROBLEMS

Many of the results above should have extensions to problems similar to those discussed in the review articles [DL, Le]. One interesting open problem is the following: In establishing blow up results, we showed (for Problem 1.1) that if $\int_{\mathbf{R}^n} w(x)dx > 0$ and $n \geq 3$ there is a finite critical exponent $p = n/(n-2)$. On the other hand, there exist global solutions for some w satisfying $\int_{\mathbf{R}^n} w(x)dx < 0$ and any $p > 1$. Suppose that $\int_{\mathbf{R}^n} w(x)dx = 0$ but $w \neq 0$. Then which situation obtains, that for which the integral is positive or that for which the integral is negative?

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